Quantitative OE & isoperimetric profile

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I Amenability

Γ countable, Γ is amenable if $\exists (F_n)$ $F_n \Subset \Gamma$, $\forall \gamma \in \Gamma$,

$$\frac{|\gamma F_n \Delta F_n|}{|F_n|} \to 0 \quad (*)$$

such a seq is a Følner sequence

Fact: If $\Gamma = \langle S \rangle$, suffices to show $(*)$ holds for $\gamma \in S$

Ex: Finite groups ($F_n = \Gamma$), loc finite groups $\Gamma = \bigcup_n \Gamma_n$, Γ_n finite $\quad F_n = \Gamma_n$

II / Growth

$\Gamma = \langle S \rangle$

S finite, $S = S^{-1}$

$$d_S(\gamma, \gamma') = \mathrm{Min}\{k \in \mathbb{N} \mid \gamma = s_1 \cdots s_k \gamma', \quad s_i \in S\}$$

growth funct° $C_{\Gamma,S}(n) = |B_{d_S}(e,n)|$

depends on S, but, not up to <u>asymptotic equivalence</u>

<u>Def</u>: $f, g : \mathbb{N} \to \mathbb{R}^+$, $f \prec g$ if $\exists\, C > 0$ / for large enough n,

$$f(n) \leq C\, g(Cn)$$

f is as. eq. to g ($f \sim g$) iff $f \prec g$ & $g \prec f$.

3 types of growth:

- polynomial: $\exists d \in \mathbb{N} \,/\, C_\Gamma(n) < n^d$

  Prop: $B_{d_S}(e,n)$ is a Følner sequence.

- Intermediate: $\dfrac{\ln(C_\Gamma(n))}{\ln n} \to +\infty$

  but $\dfrac{\ln(C_\Gamma(n))}{n} \to 0$ ex $e^{n^{3/4}}$

  Fact: $\exists$ subsequence of balls which is Følner.

- Exponential: $C_\Gamma(n) \sim e^n$ ex: lamplighter

Thm (Grigorchuk) There is a continuum of $\sim$ classes for intermediate growth.

III / Isoperimetric profile

Def: The iso profile of $\Gamma = \langle S \rangle$ is (finite symmetric $\to S$)

$$i_{\Gamma,S}(n) = \sup_{|F| \leq n} \frac{|F|}{|\partial_S F|}$$

(well def i_Γ up to $\sim$)

$$\partial_S F = \Big(\bigcup_{s \in S} sF \Big) \setminus F$$

ex: $i_{\mathbb{Z}^d}(n) = n^{1/d}$

Note: Γ is amenable iff $i_\Gamma(n) \to +\infty$

Thm (Brieussel - Zheng) There is a continuum of exp growth amenable gps with non $\sim$ iso profile

IV / Quantitative OE

Everything is measurable, act° are measure-preserving

(X,μ) stand proba space $(\cong ([0,1], \mathrm{Leb}))$

$\Gamma \curvearrowright (X,\mu)$ is <u>free</u> if $\forall \gamma \in \Gamma \setminus \{e\}$, $\mu(\{x \in X \mid \gamma x = x\}) = 0$

<u>Ex</u>: • $\Gamma \curvearrowright \underset{\substack{\shortparallel \\ \mathcal{P}(\Gamma)}}{\{0,1\}^\Gamma}$, $\mu = \left(\frac{1}{2}(\delta_0 + \delta_1)\right)^{\otimes \Gamma}$ is free

$$\gamma \cdot (x_g)_{g \in \Gamma} = (x_{\gamma^{-1} g})_{g \in \Gamma}$$

• <u>odometer</u> $\Gamma = \mathbb{Z}$, $X = \{0,1\}^{\mathbb{N}}$, $\mu = \left(\frac{1}{2}(\delta_0 + \delta_1)\right)^{\otimes \mathbb{N}}$

$\hookrightarrow T_0((x_n)) = (x_n) + (1, 0, 0, \ldots)$
with carry to the right.

ex $T_0(1,1,1,0,1,0,\dots) = (0,0,0,1,1,0,\dots\dots)$
$+(1,0\dots\dots)$

Def (Dye) Γ & Λ are orbit equivalent (OE)

if $\exists\ \Gamma, \Lambda \curvearrowright (X,\mu)$ free with same orbits: $\forall^* x \quad \Gamma x = \Lambda x$

If we define $R_\Gamma := \{(x, \gamma x) : \gamma \in \Gamma, x \in X\}$

we are asking $R_\Gamma = R_\Lambda$ up to a null

exercise: ∘ $\mathbb{Z}$ is orbit equivalente $\mathcal{P}_f(\mathbb{N})$

∘ using $\{0,1\}^{\mathbb{N}} \cong \{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}} (= \{0,1\}^{\mathbb{N}\times\mathbb{N}})$

show $\mathbb{Z}^2$ is OE to $\mathbb{Z}$

Thm (Ornstein-Weiss) All countable infinite amenable groups are OE.

$\rightsquigarrow$ Quantitative OE can distinguish amen gps
(Bader-Furman-Sauer, Austin, Bowen)

Def: $\Gamma \curvearrowright (X,\mu)$, $\Gamma = \langle S \rangle$ ← finite sym

endow X with a metric $d_S(x,y) = \inf\{k : x = s_1 \cdots s_k y, \ s_i \in S\}$

$\underset{\langle S_\Lambda \rangle}{\Lambda}, \underset{\langle S_\Gamma \rangle}{\Gamma} \curvearrowright (X,\mu)$ free OE act°

$p > 0$ this OE is L^p if : • $\forall \gamma \in \Gamma$, $\int_X (d_{S_\Lambda}(x, \gamma x))^p < +\infty$

• $\forall \lambda \in \Lambda$, $\int_X (d_{S_\Gamma}(x, \lambda x))^p < +\infty$

$\rightsquigarrow$ then Γ and Λ are L^p OE

L^1 OE is an equivalence relation

L^p OE is not but Γ L^p OE Λ L^q OE Δ

then Γ is L^{pq} OE Δ

"pseudo-metric on fin. gen. groups"

$$d(\Gamma,\Lambda) = \sup_{\substack{p\leq 1 \\ \Gamma \text{ is } L^p \text{ OE } \Lambda}} - \log p$$

<u>Thm</u> (Bowen) $p\leq 1$, Γ L^p OE Λ $\Rightarrow$ $c_\Gamma^p \leq c_\Lambda$

($p=1 \longrightarrow c_\Gamma$ is L^1 OE inv)

<u>Thm</u> (Austin) For polynomial growth gps,

the <u>asymptotic cone</u> is an L^1 OE invariant

$\rightsquigarrow$ $\mathbb{Z}^4$ and $H(\mathbb{Z}) = \left\{ \begin{pmatrix} 1 & x & y \\ & 1 & z \\ & & 1 \end{pmatrix} \right\}$ both have n^4 growth

but they are not L^1 OE because $\neq$ as cones.

<u>Thm</u> (D, K, LM, T) $\exists$ OE between $\mathbb{Z}^4$ and $H(\mathbb{Z})$

which is L^p $\forall p < 1$

Example of L^p OE, using odometer, $d, d' \geq 1$, $d < d'$

$$\mathbb{Z}^d \text{ is } L^p \text{ OE } \mathbb{Z}^{d'} \quad \forall p < d/d'$$

not possible L^p for $p > \frac{d}{d'}$
(by Bowen)

Q°: what about $p = \frac{d}{d'}$

Thm Γ L^p OE Λ

$$\Longrightarrow \quad i^p_\Gamma \leq i_\Lambda$$

B–Z: continuum of exp growth amenable gps non pairwise L^1 OE.

$\rightsquigarrow \Gamma = (\mathcal{P}_f(\mathbb{Z}), \Delta) \cong \bigoplus_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$

- $\mathbb{Z} \longrightarrow F_n = [\![-n, n]\!]$

  $\mathbb{Z}^d \longrightarrow [\![-n, n]\!]^d$

- Lamplighter group : $\mathcal{P}_f(\mathbb{Z}) \rtimes \mathbb{Z}$

  set of lamps which are on

  coordinate of lamplighter

$$(A, m) \cdot (B, n) = \big(A \,\Delta\, (m + B),\ m + n \big)$$

is gen by $s = (\{0\}, 0)$ $\quad (A, m) \cdot s = (A, m) \cdot (\{0\}, 0)$

$t = (\emptyset, 1)$ $\qquad = (A \,\Delta\, \{m\}, m)$

$(A, m) \cdot t = (A, m + 1)$

$$F_n = \mathcal{P}([\![-n, n]\!]) \times [\![-n, n]\!]$$